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COHOMOLOGICAL ASPECTS OF HOPF ALGEBRA LIFTINGS

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ABSTRACT. A recent result of ours [GM] shows that all Hopf algebra liftings of a given diagram in the sense of Andruskiewitsch and Schneider are cocycle deformations of each other. Here we develop a 'non-abelian' cohomology theory, which gives a method for an explicit description of cocycles relevant to the lifting process.

0. INTRODUCTION

The Nichols algebra $B(V)$ of a crossed kG-module V is a connected braided Hopf algebra. In terms of generators and relations it can be described via a certain pushout diagram

$$\begin{array}{ccc} K(V) & \xrightarrow{\kappa} & R(V) \\ {\scriptstyle\varepsilon}\downarrow & & {\scriptstyle\pi}\downarrow \\ k & \xrightarrow{\iota} & B(V) \end{array}$$

of connected braided Hopf algebras. The Radford biproduct or bosonization $H(V) = B(V)\#kG$ has a similar presentation in the category of ordinary Hopf algebras. A lifting of $H(V)$ is a pointed Hopf algebra H for which $\operatorname{gr}^c H \cong H(V)$, where $\operatorname{gr}^c H$ is the graded Hopd algebra associated with the coradical filtration of H. Such liftings are obtained by deforming the multiplication of $H(V)$. In this context the lifting problem for V is asking for the characterization and classification of all liftings of $H(V)$. This problem has been solved by Andruskiewitsch and Schneider in [AS] for a large class of crossed kG-modules of finite Cartan type, which will carry the attribute 'special' in this paper. This allows, in particular, for a classification of all finite dimensional pointed Hopf algebras A for which the order of the abelian group of points is not divisible by any prime < 11. In recent work [GM] we have shown that for any given V in this class all liftings of $H(V)$ are cocycle deformations of each other (see also [Ma1], Appendix). This is done via a description of the lifted Hopf algebras suitable for the application of results by Masuoka about Morita-Takeuchi equivalence [Ma] and by Schauenburg about Hopf-Galois extensions [Sch]. For some special cases such results had been obtained in [Ma, Di, BDR, Gr]. In addition, our results in [GM] show that

every lifting of $H(V)$, and therefore the corresponding cocycle, is completely determined by a G-invariant algebra map $f \in \mathrm{Alg}_G(K(V), k)$, but without an explicit description of the corresponding cocycle in terms of f.

In the present paper we aim at making this connection between the G-invariant algebra map $f \in \mathrm{Alg}_G(K, k)$ and the corresponding deforming cocycle $\sigma : B \otimes B \to k$ more explicit. For that purpose we first describe a non-abelian equivariant cohomology theory for braided Hopf algebras X in the category of crossed H-modules and for their bosonizations $X\#H$, where H is an ordinary Hopf algebra. The Radford biproduct $X\#H$ is an ordinary Hopf algebra and carries the obvious H-bimodule structure. A pushout diagram of (braided) Hopf algebras as above, in which κ has a H-module coalgebra retraction gives rise to a Meier-Vietories type 5-term exact sequence

$$1 \to \mathrm{Alg}_H(B, k) \xrightarrow{\pi^*} \mathrm{Alg}_H(R, k) \xrightarrow{\kappa^*} \mathrm{Alg}_H(K, k) \xrightarrow{\delta} \mathcal{H}^2_H(B, k) \xrightarrow{\pi^*} \mathcal{H}^2_H(R, k)$$

of pointed sets. In the situation of the lifting problem, when B is the Nichols algebra of a crossed kG-module of special finite Cartan type, then $\mathrm{Alg}_G(R, k)$ is trivial. If, in addition, K is a K-bimodule coalgebra retract in R, then the connecting map $\delta : \mathrm{Alg}_G(K(V), k) \to \mathcal{H}^2_G(B(V), k)$ exists and is injective. Then, in view of the characterization of liftings in [AS, GM], the cocycles obtained via the connecting map account for all liftings of $B(V)\#kG$.

The 5-term sequence for equivariant Hochschild cohomology

$$0 \to \mathrm{Der}_H(B, k) \xrightarrow{\pi^*} \mathrm{Der}_H(R, k) \xrightarrow{\kappa^*} \mathrm{Der}_H(K, k) \xrightarrow{\delta} H^2_H(B, k) \xrightarrow{\pi^*} H^2_H(R, k)$$

has been established in [GM] and is an exact sequence of vector spaces. Here it suffices that K is a K-bimodule retract in R, which in the liftings situation is always the case. The question about the relationship between Hochschild cohomology and non-abelian cohomology naturally arises in this context. In the cocommutative case there are Sweedler's results. For quantum linear spaces, i.e: for diagrams of type $A_1 \times A_1 \times \ldots \times A_1$, there is an exponential relationship between Hochschild cocycles and those 'multiplicative' cocycles which depend on the root vector parameters alone [GM]. Here we present some more general results on this topic involving linking as well. This includes an approach to quantum planes quite different from that of [ABM], Section 5. In the last section we also develop a program for the connected case, and apply it to diagrams of type A_2. Results for type A_n, $n > 2$, and for type B_2 will be part of a forthcoming paper.

The notation in the paper as in [GM] is pretty much standard; $m : A \otimes A \to A$ denotes multiplication, $\Delta : C \otimes C \to C$ comultiplication, $s : H \to H$ the antipode, and $* : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A)$ the convolution multiplication $f * f' = m(f \otimes f')\Delta$. We use Sweedler's notation in the form $\Delta(c) = c_1 \otimes c_2$ etc., and also $\Delta^{(n)} = (1 \otimes \Delta^{(n-1)})\Delta$ for $n \geq 1$ with $\Delta^{(0)} = 1$. The notaion used for coactions of a Hopf algebra $\delta : X \to H \otimes X$ is $\delta(x) = x_{-1} \otimes x_0$.

1. A non-abelian cohomology

Every lifting of the bosonisation $A = B\#kG$ of the Nichols algebra B of a finite dimensional special crossed G-module V is determined by a G-invariant algebra map $f \in_G \mathrm{Alg}_G(K\#kG, k)$, and it is also a cocycle deformation A_σ of A. The G-invariant 'multiplicative' cocycle $\sigma : A \otimes A \to k$ must therefore be completely determined by the G-invariant algebra map $f : K \to k$. In the examples presented in [GM] Section 3 the relation between the two entities is given explicitly. In this paper non-abelian cohomology will serve to clarify this relationship for some special diagrams of finite Cartan type.

1.1. **The 'multiplicative' cohomology.** The non-abelian equivariant cohomology of a braided Hopf algebra in the category of crossed H-modules X or its bosonization, which is an ordinary Hopf algebra, is defined via the cosimplicial group complex of regular elements

$$\mathrm{Reg}_H(k,k) \;\overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}}\; \mathrm{Reg}_H(X,k) \;\overset{\partial^0,\partial^1,\partial^2}{\longrightarrow}\; \mathrm{Reg}_H(X^2,k) \;\overset{\partial^0,\partial^1,\partial^2,\partial^3}{\longrightarrow}\; \mathrm{Reg}_H\ (X^3,k)$$

in the standard cosimplicial algebra complex

$$\mathrm{Hom}_H(k,k) \;\overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}}\; \mathrm{Hom}_H(X,k) \;\overset{\partial^0,\partial^1,\partial^2}{\longrightarrow}\; \mathrm{Hom}_H(X^2,k) \;\overset{\partial^0,\partial^1,\partial^2,\partial^3}{\longrightarrow}\; \mathrm{Hom}_H\ (X^3,k),$$

where X^i denotes the i-th tensor power of X, and where

$$\partial^i f = \begin{cases} \varepsilon \otimes f & \text{if } i = 0 \\ f(1^{i-1} \otimes m \otimes 1^{n-i-1}) & \text{if } 0 < i < n \\ f \otimes \varepsilon & \text{if } i = n \end{cases}$$

are the standard cofaces.

The first equivariant 'non-abelian' cohomology of X with coefficients in k is given by

$$\mathcal{H}^1_H(X,k) = Z^1_H(X,k) = \{f \in \mathrm{Reg}_H(X,k) | \partial^1 f = \partial^2 f * \partial^0 f\} = \mathrm{Alg}_H(X,k)$$

which is a group under the convolution multiplication. A 1-cocycle is therefore an element $f \in \mathrm{Reg}_H(X,k)$ such that $fm = (f \otimes \varepsilon) * (\varepsilon \otimes f) = m_k(f \otimes f)$, that is an algebra map. For the second cohomology define the set of 'non-abelian' 2-cocycles by

$$Z^2_H(X,k) = \{\sigma \in \mathrm{Reg}_H(X^2,k) | \partial^0\sigma * \partial^2\sigma = \partial^3\sigma * \partial^1\sigma, \sigma(\iota \otimes 1) = \varepsilon = \sigma(1 \otimes \iota)\}$$

which means that $\sigma \in \mathrm{Reg}_H(X^2,k)$ is a cocycle if and only if the 'multiplicative' 2-cocycle conditions

$$(\varepsilon \otimes \sigma) * \sigma(1 \otimes m) = (\sigma \otimes \varepsilon) * \sigma(m \otimes 1)\ ,\ \sigma(\iota \otimes 1) = \varepsilon = \sigma(1 \otimes \iota)$$

are satisfied, in particular $\sigma(y_1 \otimes z_1)\sigma(x \otimes y_2 z_2) = \sigma(x_1 \otimes y_1)\sigma(x_2 y_2 \otimes z)$ in the ordinary case and $\sigma(y_1 \otimes (y_2)_{-1} z_1)\sigma(x \otimes (y_2)_0 z_2) = \sigma(x_1 \otimes (x_2)_{-1} y_1)\sigma((x_2)_0 y_2 \otimes z)$ in the braided case. Define a relation on $\mathrm{Reg}_H(X^2, k)$ by declaring $\sigma \sim \sigma'$ if and only if $\sigma' = \partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}$ for some $\chi \in \mathrm{Reg}_H(X, k)$.

Lemma 1.1. *The relation $\sim$ just defined on $\mathrm{Reg}_H(X^2, k)$ is an equivalence relation, which restricts to $Z^2_H(X, k)$. The second "non-abelian" cohomology $\mathcal{H}^2_H(X, k) = Z^2_H(X, k)/\sim$ is a pointed set with distinguished element class$(\varepsilon\otimes\varepsilon) = \mathrm{im}(\partial : \mathrm{Reg}_H(X, k) \to \mathrm{Reg}_H(X \otimes X, k))$, where $\partial f = \partial^0 f * \partial^2 f * \partial^1 f^{-1}$. Moreover, there is a natural isomorphism $\mathcal{H}^1_H(X, k) \cong \mathcal{H}^1_H(X\#H, k)$ and a natural injection $\mathcal{H}^2_H(X, k) \to \mathcal{H}^2_H(X\#H, k)$ for braided Hopf algebras X in the category of crossed H-modules and their bosonisations $Y = X\#H$.*

Proof. First we will show that it is sufficient to prove the assertions for ordinary Hopf algebras. If X is a Hopf algebra in the category of crossed H-modules then $Y = X\#H$ is an ordinary Hopf algebra. The linear map

$$\psi_n : Y^n \to X^n$$

defined inductively by $\psi_1(xh) = x\varepsilon(h)$ and $\psi_n(xh \otimes y) = x \otimes h\psi_{n-1}y$ is a H-bimodule map (diagonal left and trivial right H-action on X^n), which has linear right inverse $\phi_n : X^n \to Y^n$ given by $\phi_1(x) = x1$ and $\phi_n(x \otimes y) = x1 \otimes \phi_{n-1}y$. It factors through $Y^{(n)} = Y \otimes_H Y \otimes_H \ldots \otimes_H Y$ to give a left H-module isomorphism $Y^{(n)} \otimes_H k \cong X^n$. Induction on n shows that it is also compatible with the 'coalgebra structures' in that $\Delta_{X^n}\psi_n = (\psi_n \otimes \psi_n)\Delta_{Y^n}$ and $\varepsilon\psi_n = \varepsilon$. The induced injective algebra map

$$\psi^n : \mathrm{Hom}_H(X^n, k) \to \mathrm{Hom}_H(Y^n, k)$$

is then given by $\psi^n(f) = f\psi_n$, that is $\psi^n(f)(xh \otimes y) = f(x \otimes h\psi_{n-1}y)$ or $\psi^n f(x^1h^1 \otimes x^2h^2 \otimes \ldots \otimes x^n g^n) = f(x^1 \otimes h^1_1 x^2 \otimes \ldots \otimes h^1_{n-1}h^2_{n-2} \ldots h^{n-1}x^n)$. It is an algebra map, since it preserves the convolution multiplication,

$$\begin{aligned}\psi^n(f * f') = (f * f')\psi_n = (f \otimes f')\Delta_{X^n}\psi_n &= (f \otimes f')(\psi_n \otimes \psi_n)\Delta_{Y^n} \\ &= (f\psi_n \otimes f'\psi_n)\Delta_{Y^n} = \psi^n(f) * \psi^n(f')\end{aligned}$$

and the convolution identity, $\psi^n(\varepsilon) = \varepsilon\psi_n = \varepsilon$. It therefore automatically restricts to an injective group homomorphism

$$\psi^n : \mathrm{Reg}_H(X^n, k) \to \mathrm{Reg}_H(Y^n, k)$$

between the groups of regular elements. This leads to a injective homomorphism of the standard cosimplicial groups

$$
\begin{array}{ccccccc}
\mathrm{Reg}_H(k,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\end{array} & \mathrm{Reg}_H(X,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\end{array} & \mathrm{Reg}_H(X^2,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\\ \xrightarrow{\partial^3}\end{array} & \mathrm{Reg}_H(X^3,k) \\
\| & & \downarrow \psi^1 & & \downarrow \psi^2 & & \downarrow \psi^3 \\
\mathrm{Reg}_H(k,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\end{array} & \mathrm{Reg}_H(Y,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\end{array} & \mathrm{Reg}_H(Y^2,k) & \begin{array}{c}\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\\ \xrightarrow{\partial^3}\end{array} & \mathrm{Reg}_H(Y^3,k)
\end{array}
$$

compatible with the standard cofaces

$$
\partial^i f = \begin{cases} \varepsilon \otimes f & \text{if } i = 0 \\ f(1^{i-1} \otimes m \otimes 1^{n-i-1}) & \text{if } 0 < i < n \\ f \otimes \varepsilon & \text{if } i = n \end{cases}
$$

in which ψ^1 is an isomorphism. It then suffices to prove the first assertion for the ordinary Hopf algebra $Y = X\#H$.

First observe that if $f \in \mathrm{Reg}_H(Y,k)$ then $\partial f = \partial^0 f * \partial^2 f * \partial^1 f^{-1}$ is a 2-cocycle:

$$
\begin{aligned}
&\big((\varepsilon \otimes \partial f) * \partial f(1 \otimes m)\big)(x \otimes y \otimes z) \\
&\quad= \partial f(y_1 \otimes z_1)\partial f(x \otimes y_2 z_2) \\
&\quad= f(z_1)f(y_1)f^{-1}(y_2 z_2)f(y_3 z_3)f(x_1)f^{-1}(x_2 y_4 z_4) \\
&\quad= f(x_1)f(y_1)f(z_1)f^{-1}(x_2 y_2 z_2) \\
&\quad= f(y_1)f(x_1)f^{-1}(x_2 y_2)f(z_1)f(x_3 y_3)f^{-1}(x_4 y_4 z_2) \\
&\quad= \partial f(x_1 \otimes y_1)\partial f(x_2 y_2 \otimes z) \\
&\quad= (\partial f \otimes \varepsilon) * \partial f(m \otimes 1)(x \otimes y \otimes z)
\end{aligned}
$$

Now we show that $\sim$ is an equivalence relation even on $\mathrm{Reg}_H(Y,k)$, and that it restricts to $Z^2_H(Y,k)$.

Reflexivity, $\sigma \sim \sigma$ of the relation $\sim$ obviously holds with $\chi = \varepsilon$.

To check symmetry, observe that $\sigma' = \partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}$ for some $\chi \in \mathrm{Reg}_H(Y,k)$ implies that $\sigma = \partial^0\chi^{-1} * \partial^2\chi^{-1} * \sigma' * \partial^1\chi$ since $(\partial^i\chi)^{-1} = \partial^i\chi^{-1}$ and $\partial^2\chi * \partial^0\chi = \partial^0\chi * \partial^2\chi$.

For transitivity suppose that in addition $\sigma'' = \partial^0\psi * \partial^2\psi * \sigma' * \partial^1\psi^{-1}$ for some $\psi \in \mathrm{Reg}_H(Y,k)$. Then

$$
\begin{aligned}
\sigma'' &= \partial^0\psi * \partial^2\psi * \partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1} * \partial^1\psi^{-1} \\
&= \partial^0(\psi * \chi) * \partial^2(\psi * \chi) * \sigma * \partial^1(\psi * \chi)^{-1}
\end{aligned}
$$

since $\partial^2\psi * \partial^0\chi = \partial^0\chi * \partial^2\psi$ and the ∂^i are group homomorphisms.

To show that the equivalence relation $\sim$ restricts to $Z^2_H(Y,k)$ it suffices to show that if $\sigma \in Z^2_H(Y,k)$ and $\chi \in \mathrm{Reg}_H(Y,k)$ then $\sigma' = \partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}$ is a

cocycle as well:

$$\begin{aligned}
&\big(\partial^0\sigma' * \partial^2\sigma'\big)(x\otimes y\otimes z) = \sigma'(y_1\otimes z_1)\sigma'(x\otimes y_2z_2)\\
&= \chi(z_1)\chi(y_1)\sigma(y_2\otimes z_2)\chi^{-1}(y_3z_3)\chi(y_4z_4)\chi(x_1)\sigma(x_2\otimes y_5z_5)\chi^{-1}(x_3y_6z_6)\\
&= \chi(x_1)\chi(y_1)\chi(z_1)\sigma(y_2\otimes z_2)\sigma(x_2\otimes y_3z_3)\chi^{-1}(x_3y_4z_4)\\
&= \chi(x_1)\chi(y_1)\chi(z_1)\sigma(x_2\otimes y_2)\sigma(x_3y_3\otimes z_2)\chi^{-1}(x_4y_4z_3)\\
&= \chi(y_1)\chi(x_1)\sigma(x_2\otimes y_2)\chi^{-1}(x_3y_3)\chi(z_1)\chi(x_4y_4)\sigma(x_5y_5\otimes z_2)\chi^{-1}(x_6y_6z_3)\\
&= \sigma'(x_1\otimes y_1)\sigma'(x_2y_2\otimes z) = (\partial^3\sigma' * \partial^1\sigma')(x\otimes y\otimes z)
\end{aligned}$$

and

$$\sigma'(x\otimes 1) = \chi(x_1)\sigma(x_2\otimes 1)\chi^{-1}(x_3) = \varepsilon(x) = \chi(x_1)\sigma(1\otimes x_2)\chi^{-1}(x_3) = \sigma'(1\otimes x).$$

This proves the assertions for the bosonisation $Y = X\#H$. For the braided Hopf algebra X they are now a consequence of the properties of the diagram above. It follows that

$$\mathrm{Alg}_H(X,k) = \mathcal{H}^1_H(X,k) \cong \mathcal{H}^1_H(Y,k) = \mathrm{Alg}_H(Y,k)$$

since ψ^1 is an isomorphism and ψ^2 is injective. Since, in addition, ψ^3 is injective as well it follows that $\sigma \in \mathrm{Reg}_H(X^2,k)$ is a 2-cocycle if and only if $\psi^2\sigma \in \mathrm{Reg}_H(Y{,}k)$ is a cocycle. In particular, if $f \in \mathrm{Reg}_H(X,k)$ then $\partial f = \partial^0 f * \partial^2 f * \partial^1 f^{-1} \in Z^2(X,k)$. Moreover, the following argument shows that the induced map $\mathcal{H}^2_H(X,k) \to \mathcal{H}^2(Y,K)$ is injective. Suppose that $\sigma,\sigma' \in Z^2(X,k)$ are such that $\psi^2\sigma \sim \psi^2\sigma'$ in $Z^2(Y,k)$. This means that

$$\psi^2\sigma' = \partial^0\psi^1\phi * \partial^2\psi^1\phi * \psi^2\sigma * \partial^1\psi^1\phi^{-1} = \psi^2(\partial^0\phi * \partial^2\phi * \sigma * \partial^1\phi^{-1})$$

for some $\phi \in \mathrm{Reg}_H(X,k)$, and hence $\sigma' = \partial^0\phi * \partial^2\phi * \sigma * \partial^2\phi^{-1}$, where we used the fact that ψ^1 is an isomophism and ψ^2 is an injective algebra map. $\square$

Our aim here is to describe cocycle deformations of the bosonizations $Y = X\#H$ of braided Hopf algebras X in the category of crossed H-modules. The following calculation shows that equivalent cocycles lead to isomorphic deformations.

Proposition 1.2. *Let $Y = X\#H$ be the bosonization of a braided Hopf algebra X in the category of crossed H-modules. If $\sigma,\sigma' \in Z^2_H(Y,k)$ are in the same cohomology class then the cocycle deformations Y_σ and $Y_{\sigma'}$ are isomorphic.*

Proof. Suppose that $\sigma' = \partial^0\chi*\partial^2\chi*\sigma*\partial^1\chi^{-1}$ for some $\chi \in \mathrm{Reg}_H(X,k)$. It suffices to show that the equivariant coalgebra automorphism $\psi = \chi^{-1} * 1 * \chi : Y \to Y$ is actually also an algebra map $\psi : Y_\sigma \to Y_{\sigma'}$. And it is, since

$$\begin{aligned}
&m_{\sigma'}(\psi x\otimes\psi y) = \chi^{-1}(x_1)\chi^{-1}(y_1)m_{\sigma'}(x_2\otimes y_2)\chi(x_3)\chi(y_3)\\
&= \chi^{-1}(x_1)\chi^{-1}(y_1)\sigma'(x_2\otimes y_2)x_3y_3{\sigma'}^{-1}(x_4\otimes y_4)\chi(x_5)\chi(y_5)\\
&= \sigma(x_1\otimes y_1)\chi^{-1}(x_2y_2)x_3y_3\chi(x_4y_4)\sigma^{-1}(x_5\otimes y_5)\\
&= \sigma(x_1\otimes y_1)\psi(x_2y_2)\sigma^{-1}(x_3\otimes y_3)\\
&= \psi m_\sigma(x\otimes y)
\end{aligned}$$

implies that $\psi m_{\sigma'} = m_\sigma(\psi \otimes \psi)$. $\square$

2. A 5-term sequence in 'non-abelian' cohomology

A commutative 'pushout' square of (braided) Hopf algebras in the introduction and its bosonisation can help to get an explicit description of the deforming cocycles σ on B and of the corresponding cocycles on the bosonization $A = B\#H$ in terms of the H-invariant algebra maps $f \in \mathrm{Alg}_H(K,k)$, , Such squares of (braided) Hopf algebras

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & R \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ k & \xrightarrow{\iota} & B \end{array} \qquad\qquad \begin{array}{ccc} K\#H & \xrightarrow{\kappa\#1} & R\#H \\ {\scriptstyle\varepsilon\#1}\big\downarrow & & {\scriptstyle\pi\#1}\big\downarrow \\ H & \xrightarrow{\iota\#1} & B\#H \end{array}$$

induce a square of cosimplicial groups

$$\begin{array}{ccccccc}
\mathrm{Reg}_H(k,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}} & \mathrm{Reg}_H(B,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}} & \mathrm{Reg}_H(B^2,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\\ \xrightarrow{\partial^3}} & \mathrm{Reg}_H(B^3,k) \\
\| & & \downarrow \pi^* & & \downarrow (\pi^2)^* & & \downarrow (\pi^3)^* \\
\mathrm{Reg}_H(k,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}} & \mathrm{Reg}_H(R,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}} & \mathrm{Reg}_H(R^2,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\\ \xrightarrow{\partial^3}} & \mathrm{Reg}_H(R^3,k) \\
\| & & \downarrow \kappa^* & & \downarrow (\kappa^2)^* & & \downarrow (\kappa^3)^* \\
\mathrm{Reg}_H(k,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}} & \mathrm{Reg}_H(K,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}} & \mathrm{Reg}_H(K^2,k) & \substack{\xrightarrow{\partial^0}\\ \xrightarrow{\partial^1}\\ \xrightarrow{\partial^2}\\ \xrightarrow{\partial^3}} & \mathrm{Reg}_H(K^3,k)
\end{array}$$

where the trivial part has been omitted, and a similar square for the bosonisation. The natural injective group homomorphism $\psi : \mathrm{Reg}_H(X,k) \to \mathrm{Reg}_H(X\#H,k)$ induces a natural map between these squares. Here is a 5-term sequence for non-abelian cohomology in case $\kappa : K \to R$ has a K-bimodule coalgebra retraction $u : R \to K$.

Theorem 2.1. *If $\kappa K \to R$ has a K-bimodule coalgebra retraction then there is an exact sequences of pointed sets*

$$1 \to \mathrm{Alg}_H(B,k) \xrightarrow{\pi^*} \mathrm{Alg}_H(R,k) \xrightarrow{\kappa^*} \mathrm{Alg}_H(K,k) \xrightarrow{\delta} \mathcal{H}^2_H(B,k) \xrightarrow{\pi^*} \mathcal{H}^2_H(R,k)$$

and an injective map induced by the cosimplicial group homomorphism ψ^* into a similar exact sequence involving the bosonisations. The connecting map $\delta : \mathrm{Alg}_H(K,k) \to \mathcal{H}^2_G(B,k)$ does not depend on the particular choice of the K-bimodule coalgebra retraction $u : R \to K$.

Proof. It is clear that $\pi^* : \mathrm{Alg}_H(B,k) \to \mathrm{Alg}_H(R,k)$ is injective and that $\kappa^*\pi^* = (\pi\kappa)^* = (\iota\varepsilon)^* = \varepsilon^*\iota^*$ is the trivial map. Moreover, if $\kappa^*(f) = \varepsilon$ for $f \in \mathrm{Alg}_H(R,k)$ then, by the pushout property, there is a unique $f' \in \mathrm{Alg}_H(B,k)$ such that $\pi^*(f') = f$. To construct $\delta : \mathrm{Alg}_H(K,k) \to \mathcal{H}^2_H(B,k)$ observe first that

$$\begin{aligned}\mathrm{Alg}_H(K,k) &= Z^1_H(K,k) = \mathcal{H}^1_H(K,k) = \{f \in \mathrm{Reg}_H(K,k) | \partial^1 f = \partial^2 f * \partial^0 f\} \\ &= \{f \in \mathrm{Reg}_H(K,k) | \partial^0 f * \partial^2 f * \partial^1 fs = \varepsilon \otimes \varepsilon\}.\end{aligned}$$

The existence of a H-invariant K-module coalgebra retraction $u : R \to K$ for the injection $\kappa : K \to R$ implies that, for every $f \in \mathrm{Alg}_H(K,k)$, the map $fu \in \mathrm{Hom}_H(R,k)$ is convolution invertible with inverse fsu. Then by Lemma 1.1 the map

$$\sigma_R = \partial u^* f = \partial^0 fu * \partial^2 fu * \partial^1 fsu : R \otimes R \to k$$

is a convolution invertible 2-cocycle with inverse $\sigma_R^{-1} = \partial^1 fu * \partial^2 fsu * \partial^0 fsu$, in particular $\sigma_R(x \otimes y) = fu(x_1)fu(y_1)fsu(x_2y_2)$. It satisfies the 2-cocycle conditions

$$\sigma_R(1 \otimes \iota) = \varepsilon = \sigma_R(\iota \otimes 1)) \ , \ (\varepsilon \otimes \sigma_R) * \sigma_R(1 \otimes m) = (\sigma_R \otimes \varepsilon) * \sigma_R(m \otimes 1).$$

Now $(\kappa \otimes \kappa)^* \partial^i u^* = \partial^i \kappa^* u^* = \partial^i$ for $i = 0, 1, 2$, so that $(\kappa \otimes \kappa)^* \partial fu = \partial f = \varepsilon \otimes \varepsilon$, since $f : K \to k$ is an algebra map. Moreover, because u is a H-invariant K-bimodule coalgebra map and $f : K \to k$ is a H-invariant algebra map it follows that $(fu \otimes 1)c = (fu \otimes 1)\tau$ and $fm_K = f \otimes f$, so that

$$\begin{aligned}\partial u^* f &= (\varepsilon \otimes fu \otimes fu \otimes \varepsilon \otimes fsum)(\Delta_{R\otimes R} \otimes 1 \otimes 1)\Delta_{R\otimes R} \\ &= ((fu \otimes fu)c \otimes fum)\Delta_{R\otimes R} = (fu \otimes fu \otimes fsum)\Delta_{R\otimes R} \\ &= (fu \otimes fu \otimes fsum)(1 \otimes c \otimes 1)(\Delta_R \otimes \Delta_R) \\ &= (fu \otimes fu \otimes fsum)(1 \otimes \tau \otimes 1)(\Delta_R \otimes \Delta_R)\end{aligned}$$

and $\partial fu(xr \otimes r') = \varepsilon(x)\partial fu(r \otimes r') = \partial fu(r \otimes r'x)$ for all $x \in K$ and $r, r' \in R$, which says that $\partial fu : R \otimes R \to k$ is a K-bimodule map. This means in particular that

$$\partial u^* f(K^+ R \otimes R + R \otimes RK^+) = 0$$

and hence that the cocycle $\sigma_R = \partial u^* f : R \otimes R \to k$ factors uniquely through $\pi \otimes \pi : R \otimes R \to B \otimes B$, i.e: there exists a unique $\sigma : B \otimes B \to k$ such that $(\pi \otimes \pi)^* \sigma = \partial u^* f$. Since $\pi : R \to B$ is a surjective Hopf algebra map, this $\sigma : B \otimes B \to k$ is a 2-cocycle as well. So define

$$\delta : \mathrm{Alg}_H(K,k) \to Z^2_H(B,k)$$

by $\delta(f) = \sigma$.

Exactness at $\mathrm{Alg}_H(K,k)$: If $f \in \mathrm{Alg}_H(K,k)$ and $\delta f = \partial\chi$ for some $\chi \in \mathrm{Reg}_H(B,k)$ then $\partial fu = (\pi \otimes \pi)^* \partial\chi = \partial\pi^*\chi$ and $g = \pi^*\chi^{-1} * fu \in \mathrm{Reg}_H(R,k)$ and $\kappa^* g = \kappa^*(\chi^{-1}\pi * fu) = \chi^{-1}\pi\kappa * f\kappa u = \chi^{-1}\iota\varepsilon * f = \varepsilon * f = f$. It remains to

show that $g \in \mathrm{Alg}_H(R,k)$. But $\partial g = \varepsilon \otimes \varepsilon$, since

$$\begin{aligned}
\partial g &= \partial^0 g * \partial^2 g * \partial^1 g^{-1} \\
&= \partial^0(\chi^{-1}\pi * fu)) * \partial^2(\chi^{-1}\pi * fu) * \partial^1(fsu * \chi\pi) \\
&= \partial^0\chi^{-1}\pi * \partial^0 fu * \partial^2\chi^{-1}\pi * \partial^2 fu * \partial^1 fsu * \partial^1\chi\pi \\
&= \partial^0\chi^{-1}\pi * \partial^2\chi^{-1}\pi * \partial^0 fu * \partial^2 fu * \partial^1 fsu * \partial^1\chi\pi \\
&= \partial^0\chi^{-1}\pi * \partial^2\chi^{-1}\pi * \partial fu * \partial^1\chi\pi \\
&= \partial^0\chi^{-1}\pi * \partial^2\chi^{-1}\pi * \partial\chi\pi * \partial^1\chi\pi \\
&= \varepsilon \otimes \varepsilon,
\end{aligned}$$

as $\partial^0 f' * \partial^2 f'' = (f' \otimes f'')c = (f' \otimes f'')\tau = f'' \otimes f' = \partial^2 f'' * \partial^0 f'$ for $f' \in \mathrm{Reg}_H(R,k)$, so that g is an algebra map.

Conversely, if $f \in \mathrm{Alg}_H(R,k)$ then $\kappa^* f \in \mathrm{Alg}_H(K,k)$, $\partial f\kappa u \in Z^2_H(R,k)$, $\delta f\kappa \in Z^2_H(B,k)$ and $(\pi \otimes \pi)^* \delta\kappa^*(f) = \partial f\kappa u$. Moreover,

$$\begin{aligned}
(f\kappa u * fs)(r\kappa(x)) &= f\kappa u(r_1(r_2)_{-1}\kappa(x_1))fs((r_2)_0\kappa(x_2)) \\
&= f\kappa u(r_1)f\kappa((r_2)_{-1}x_1)fs((r_2)_0\kappa x_2) \\
&= f\kappa u(r_1)f\kappa(x_1)fs\kappa(x_2)fs(r_2) \\
&= \varepsilon(x)(f\kappa u * fs)(r)
\end{aligned}$$

for $r \in R$ and $x \in K$, in particular $(f\kappa u * fs)(RK^+) = 0$. Hence, there is a unique $\chi \in \mathrm{Reg}_H(B,k)$ such that $f\kappa u * fs = \chi\pi$, and observe that χ is convolution invertible since $(f\kappa u * fs)^{-1}(RK^+) = (f * fs\kappa u)(RK^+) = 0$ as well. This implies that $f\kappa u = \chi\pi * f$ and

$$\begin{aligned}
\partial f\kappa u &= \partial(\chi\pi * f) = \partial^0(\chi\pi * f) * \partial^2(\chi\pi * f) * \partial^1(\chi\pi * f)^{-1} \\
&= \partial^0\chi\pi * \partial^0 f * \partial^2\chi\pi * \partial^2 f * \partial^1 fs * \partial^1\chi^{-1}\pi \\
&= \partial^0\chi\pi * \partial^2\chi\pi * \partial^0 f * \partial^2 f * \partial^1 fs * \partial^1\chi^{-1}\pi \\
&= \partial^0\chi\pi * \partial^2\chi\pi * \partial f * \partial^1\chi^{-1}\pi \\
&= \partial\chi\pi,
\end{aligned}$$

so that $(\pi \otimes \pi)^*\delta f\kappa = \partial f\kappa u = \partial\pi^*\chi = (\pi \otimes \pi)^*\partial\chi$ and $\delta\kappa^* f = \delta f\kappa = \partial\chi$, which is equvalent to $\varepsilon \otimes \varepsilon$ under the equivalence relation on $Z^2_H(B,k)$.

Exactness at $\mathcal{H}^2_H(B,k)$: If $f \in \mathrm{Alg}_H(K,k)$ then $(\pi \otimes \pi)^*\delta f = \partial fu$, which is equivalent to $\varepsilon \otimes \varepsilon$ in $Z^2_H(R,k)$.

Conversely, if $\sigma \in Z^2_H(B,k)$ and $(\pi \otimes \pi)^*\sigma = \partial f$ for some $f \in \mathrm{Reg}_H(R,k)$ then $\partial\kappa^* f = (\kappa \otimes \kappa)^*\partial f = (\pi\kappa \otimes \pi\kappa)^*\sigma = \varepsilon \otimes \varepsilon$, so that $\kappa^* f = f\kappa \in \mathrm{Alg}_H(K,k)$, $\partial f\kappa u \in Z^2_H(R,k)$ and $\delta(f\kappa) \in Z^2_H(B,k)$. It suffices to prove that $\delta f\kappa$ is equivalent to σ in $Z^2_H(B,k)$. Now, since $\partial f(RK^+ \otimes R + R \otimes RK^+) = 0$ it follows that $\partial f(r \otimes \kappa(x)) = \varepsilon(x)\partial f(r \otimes 1) + \partial f(r \otimes (\kappa(x) - \varepsilon(x))) = \varepsilon(x)\partial f(r \otimes 1) = (\varepsilon \otimes \varepsilon)(r \otimes \kappa(x))$, which implies that

$$f(r\kappa(x)) = \partial^1 f(r \otimes \kappa(x)) = \partial^2 f * \partial^0 f(r \otimes \kappa(x)) = f(r)f\kappa(x)$$

for all $r \in R$ and $x \in K$. Then $(f\kappa su * f)(RK^+) = 0$, since

$$\begin{aligned}(f * f\kappa su)(r\kappa(x)) &= f(r_1(r_2)_{-1}\kappa(x_1))f(\kappa su(r_2)_0\kappa(x_2))\\ &= f(r_1)f((r_2)_{-1}\kappa(x_1))f\kappa s(x_2)f\kappa su((r_2)_0)\\ &= f(r_1)f\kappa(x_1)f\kappa s(x_2)f\kappa su(r_2)\\ &= \varepsilon(x)(f * f\kappa su)(r),\end{aligned}$$

and hence there is a unique $\chi \in \mathrm{Reg}_H(B,k)$ such that $f * f\kappa su = \pi^*\chi$, that is $f = \chi\pi * f\kappa u$. Then

$$\begin{aligned}(\pi \otimes \pi)^*\sigma &= \partial f = \partial^0(\chi\pi * f\kappa u) * \partial^2(\chi\pi * f\kappa u)\partial^1(\chi\pi * f\kappa u)^{-1}\\ &= \partial^0\chi\pi * \partial^0 f\kappa u * \partial^2\chi\pi * \partial^2 f\kappa u * \partial^1 f\kappa su * \partial^1\chi^{-1}\pi\\ &= \partial^0\chi\pi * \partial^2\chi\pi * \partial^0 f\kappa u * \partial^2 f\kappa u * \partial^1 f\kappa su * \partial^1\chi^{-1}\pi\\ &= \partial^0\chi\pi * \partial^2\chi\pi * \partial f\kappa u * \partial^1\chi^{-1}\pi\\ &= (\pi \otimes \pi)^*(\partial^0\chi * \partial^2\chi * \delta f\kappa * \partial^1\chi^{-1}),\end{aligned}$$

since $\partial^0 f\kappa u * \partial^2\chi\pi = \partial^2\chi\pi * \partial^0 f\kappa u$, and thus

$$\sigma = \partial^0\chi * \partial^2\chi * \delta f\kappa * \partial^1\chi^{-1},$$

so that σ is equivalent to $\delta f\kappa$ in $Z^2_H(B,k)$. The remaining assertions are now obvious.

Similar and somewhat simpler arguments lead to an exact sequence of pointed sets

$$\mathcal{H}^1_H(B\#H,k) \xrightarrow{\pi^*} \mathcal{H}^1_H(R\#H,k) \xrightarrow{\kappa^*} \mathcal{H}^1_H(K\#H,k) \xrightarrow{\delta} \mathcal{H}^2_H(B\#H,k) \xrightarrow{\pi^*} \mathcal{H}^2_H(R\#H,k)$$

for the bosonisations, and the map ψ^* of cosimplicial groups induces an injective map between the two sequences. As an alternative, given the exact sequence for the bosonisations and the map ψ^* the sequence for the braided square also follows directly.

It remains to show that any two K-bimodule coalgebra retractions $u, u' : K \to R$ lead to the same connecting map $\delta : \mathrm{Alg}_H(K,k) \to \mathcal{H}^2_H(B,k)$. Observe that $\ker\pi = K^+R + RK^+$. For $f \in \mathrm{Alg}_H(K,k)$ let $\sigma, \sigma' \in Z^2_H(B,k)$ be such that $(\pi\otimes\pi)^*\sigma = \partial fu$ and $(\pi\otimes\pi)^*\sigma' = \partial fu'$. If $x \in K^+$ and $r \in R$ then $\Delta(xr) = x_1(x_2)_{-1}r_r \otimes (x_2)_0 r_2$ and

$$\begin{aligned}fu' * fsu(xr) &= fu'(x_1(x_2)_{-1}r_1)fsu((x_2)_0 r_2)\\ &= f(x_1)f((x_2)_{-1}u(r_1))fs((x_2)_0)fsu(r_2)\\ &= \varepsilon(x)fu'(r_1)fsu(r_2) = 0\end{aligned}$$

and a similar argument shows that $fu' * fsu(rx) = 0$, so that $\chi = fu' * fsu \in \mathrm{Reg}_H(B,k)$. Moreover, since the faces $\partial^i : \mathrm{Reg}_H(R,k) \to \mathrm{Reg}_H(R\otimes R,k)$ are

group homomorphisms and since $\partial^0 f' * \partial^2 f'' = \partial^2 f'' * \partial^0 f'$ it follows that

$$\begin{aligned} & l(\pi\otimes\pi)^*(\partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}) \\ = \;& \partial^0(fu' * fsu) * \partial^2(fu' * fsu) * \partial fu * \partial^1(fu' * fsu) \\ = \;& \partial^0 fu' * \partial^2 fu' * \partial^1 fsu' = (\pi\otimes\pi)^*\sigma'. \end{aligned}$$

But $(\pi\otimes\pi)^* : \mathrm{Reg}_H(B\otimes B, k) to \,\mathrm{Reg}_H(R\otimes R, k)$ is injective, so that $\partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1} = \sigma'$, which means that σ and σ' are in the same cohomology class. $\square$

Remark. For Hochschild cohomology, which in some cases can be viewed as the infinitesimal part of the 'multiplicative' cohomology, such a sequence (now of vector spaces) also exists [GM]. The proofs are similar but somewhat simpler in that case, and the requirement that the retraction $u : R \to K$ be a coalgebra map is not needed.

3. Applications to the lifting process

Every lifting of a given diagram of special finite Cartan type is by [GM] a cocycle deformation of the bosonisation $B(V)\#kG$ of the Nichols algebra $B(V)$ and is completely determined by a G-invariant algebra map $f : K(V) \to k$. In the presence of a $K(V)$-module coalgebra retraction $u : R(V) \to K(V)$ for the injection $\kappa : K(V) \to R(V)$ the deforming cocycle can be determined via the connecting map $\delta : \mathrm{Alg}_G(K, k) \to \mathcal{H}^2_G(R, k)$ described in the last section. Observe that in our case, $\mathrm{Alg}_G(B, k) = \mathrm{Alg}_G(R, k) = \{\varepsilon\}$, and that δ is injective. The simple root vectors x_α, where $\alpha \in \Phi^+$ is a simple root, generate R as an algebra. Moreover, $f(x_\alpha) = f(gx_\alpha) = \chi_\alpha(g)f(x_\alpha)$ for every $g \in G$ and $f \in \mathrm{Alg}_G(R, k)$. It follows that $\mathrm{Alg}_G(R, k) = \{\varepsilon\}$, since $q_\alpha = \chi_\alpha(g_\alpha)$ is a non-trivial root of unity for every simple root α.

By [GM] Theorem 2.2 ([AS], Theorem 2.6) it follows that the map $\vartheta : R \to B\otimes K$, given by $\vartheta(x^a z^{a'}) = x^a \otimes z^{a'}$, is a K-module isomorphism. The K-bimodule retraction

$$u = (\varepsilon\otimes 1)\vartheta : R \to K$$

for the injection $\kappa : K \to R$ has kernel B^+R and is a K-bimodule map.

3.1. **Type A_1.** In this case the retraction $u : R \to K$ is a K-module coalgebra map, since the obvious injection $v : B \to R$ is a coalgebra map, so that B^+R is a coideal in R. The injective map

$$\delta : \mathrm{Alg}_G(K, k) \to \mathcal{H}^2_G(B, k)$$

is given by $\sigma = \delta f = (fu\otimes fu) * fsum(v\otimes v) = fsum(v\otimes v)$, that is $\sigma(x^i\otimes x^j) = fsu(x^{i+j})$ and $\sigma^{-1}(x^i\otimes x^j) = fu(x^{i+j})$ for $0 \le i, j < N$. Using

$$\Delta(x^m\otimes x^n) = \sum_{0\le i\le m;0\le j\le n} \binom{m}{i}_q \binom{n}{j}_q x^i g^{m-i}\otimes x^j g^{n-j}\otimes x^{m-i}\otimes x^{n-j}$$

and the identity

$$\sum_{i+j=r} \binom{\mathrm{m}}{i}_q \binom{n}{j}_q q^{j(m-i)} = \binom{m+n}{r}_q = 1$$

of [Ka] it follows that

$$m_\sigma(x^m \otimes x^n) = \begin{cases} x^{m+n}, & \text{if } m+n < N \\ fs(z)x^{m+n-N}(1-g^N), & \text{if } m+n \geq N \end{cases}$$

3.2. **Quantum planes.** The general quantum plane $V = kx_1 \oplus kx_2$ has G-coaction $\delta(x_i) = g_i \otimes x_i$ and G-action $gx_i = \chi_i(g)x_i$, where $\chi_1(g_2)\chi_2(g_1) = 1$ and $q = \chi_1(g_1)$ is a primitive root of unity of order N. Moreover, $\chi_i^N = \varepsilon = \chi_1\chi_2$, so that $\chi_1(g_i) = q$ and $\chi_2(g_i) = q^{-1}$. In the free Hopf algebra $k < x_1, x_2 >$ the relation $x_2x_1 = qx_1x_2 + z_{21}$, where $z_{21} = [x_2, x_1] = x_2x_1 - qx_1x_2$, can be used to construct a PBW-basis. The following Lemma, which will also be used later, is helpful in this connection.

Lemma 3.1. *For a quantum plane with linkable vertices, i.e: with $\chi_1\chi_2 = \varepsilon$, the relations*

$$x_2^m x_1^n = \sum_{r=0}^{l} q^{(m-r)(n-r)} r!_q \binom{m}{r}_q \binom{n}{r}_q x_1^{n-r} x_2^{m-r} z_{21}^r + p_{mn}$$

hold in $k < x_1, x_2 >$, where $l = \min\{m, n\}$ and p_{mn} is an element in the ideal generated by $[x_1, z_{21}]$ and $[x_2, z_{21}]$.

Proof. Since the vertices are linkable, we have $z_{21}x_i = x_iz_{21} - [x_i, z_{21}]$. It follows by induction on m that

$$x_2^m x_1 = q^m x_1 x_2^m + m_q x_2^{m-1} z_{21} - \sum_{i=1}^{m-1} q^i x_2^{m-1-i}[x_2^i, z_{21}]$$

where $[x_2^i, z_{21}] = \sum_{k=1}^{i-1} x^{i-k}[x_2, z_{21}]x_2^{k-1}$, and then, if $m \geq n$, by induction on n

$$x_2^m x_1^n = \sum_{r=0}^{l} q^{(m-r)(n-r)} r!_q \binom{m}{r}_q \binom{n}{r}_q x_1^{n-r} x_2^{m-r} z_{21}^r + p_{mn},$$

where $p_{m(n+1)} = p_{mn}x_1 + \sum_{r=0}^{n} q^{(m-r)(n-r)} r!_q \binom{m}{r}_q \binom{n}{r}_q (x_1^{n-r} x_2^{m-r}[x_1, z_{21}^r] + p_{(m-r)1} z_{21}^r)$ and $p_{m1} = \sum_{i=1}^{m-1} q^i x_2^{m-i}[x_2^i, z_{21}]$. Here we used the identities $\binom{m}{r-1}\frac{(m-r+1)_q}{r_q} = \binom{m}{r}_q$ and $\binom{n}{r}_q + q^{n+1-r}\binom{n}{r-1}_q = \binom{n+1}{r}_q$.

On the other hand, by induction on n we get

$$x_2 x_1^n = q^n x_1^n x_2 + n_q x_1^{n-1} z_{21} - \sum_{i=1}^{n-1} (n-i)_q x_1^{n-i-1}[x_1, z_{21}]x_1^{i-1}$$

and then, if $m \leq n$, by induction on m

$$x_2^m x_1^n = \sum_{r=0}^{m} q^{(m-r)(n-r)} r!_q \binom{m}{r}_q \binom{n}{r}_q x_1^{n-r} x_2^{m-r} z_{21}^r - {p'}_{mn}$$

with ${p'}_{(m+1)n} = x_2 {p'}_{mn} + \sum_{r=0}^{m} q^{(m-r)(n-r)} r!_q \binom{m}{r}\binom{n}{r}\big((n-r)_q x_1^{n-r-1}[x_2^{m-r}, z_{21}] + {p'}_{1(n-r)} x_2^{m-r}\big) z_{21}^r$ and ${p'}_{1n} = \sum_{i=1}^{n-1} (n-i)_q x_1^{n-i-1}[x_1, z_{21}] x_1^{i-1}$. Here the identities $\binom{n}{r-1}_q \frac{(n+1-r)_q}{r_q} = \binom{n}{r}_q$ and $\binom{m}{r}_q + q^{m+1-r}\binom{m}{r-1}_q = \binom{m+1}{r}_q$ were used. □

The elements $x_i^N = z_i$ and $[x_2, x_1] = z_{21}$ are primitive in $k < x_1, x_2 >$, and so are $[x_1, z_2]$, $[x_2, z_1]$ and $[z_i, z_{21}]$ for $i = 1, 2$. The ideal generated by the elements $[x_1, z_2]$, $[x_2, z_1]$, $[z_1, z_{21}]$ and $[z_2, z_{21}]$ in the braided Hopf algebra $k < x_1, x_2 >$ is therefore a Hopf ideal, so that

$$R = k < x_1, x_2 > /([x_1, z_2], [x_2, z_1], [z_1, z_{21}], [z_2, z_{21}])$$

is a Hopf algebra in the category of crossed kG-modules. It follows from the Lemma above that $[z_2, z_1] = z_2 z_1 - z_1 z_2 = 0$. Thus, if K is the Hopf subalgebra of R generated by z_1, z_2 and z_{21}, then $K = k[z_1, z_2, z_{21}]$ as an algebra, and

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & R \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\pi} \\ k & \xrightarrow{\iota} & B \end{array}$$

is a pushout square of braided Hopf algebras, where $B = k < x_1, x_2 > /(z_1, z_2, z_{21})$ is the Nichols algebra of the quantum plane. By the Lemma above $R \cong (B \otimes K) \oplus J$ as a vector space, where J is the ideal in R generated by $[x_1, z_{21}]$ and $[x_2, z_{21}]$, which is not a Hopf ideal.

Proposition 3.2. *For the quantum plane the injection $\kappa : K \to R$ has a K-bimodule coalgebra retraction $u : R \to K$ defined by $u(x^a z^b + J) = \varepsilon(x^a) z^b$.*

Proof. It is clear that the lineaer map $u : R \to K$ defined by $u(x^a z^b + J) = \varepsilon(x^a) z^b$ satisfies $u\kappa = 1_K$. It is a K-bimodule map, since in R we have $[x_i, z_j] = 0$ and $[x_i, z_{21}] \in J$. It is also a coalgebra map, since its kernel $\ker u = (B^+ \otimes K) \oplus J$ is a coideal. □

Theorem 2.1 is therefore applicable and, since $\mathrm{Alg}_G(R, k) = \{\varepsilon\}$, it follows that the connecting map $\delta : \mathrm{Alg}_G(K, k) \to \mathcal{H}^2_G(B, k)$ is injective (see proof of Proposition 3.7). It is determined by $(\pi \otimes \pi)^* \delta f = \partial(fu) = (fu \otimes fu) * fsum_R$ and, since the obvious injection $v : B \to R$ is a coalgebra map, we see that

$$\sigma(x^a \otimes x^b) = \partial(fu)(x^a \otimes x^b) = fsu(x^a x^b)$$

for $0 \leq a_i, b_i < N$, where the cocycle $\sigma = \partial(fu)(v \otimes v)$ represents the cohomology class $\delta f \in \mathcal{H}^2_G(B,k)$. In particular, in view of the definition of u and Lemma 3.1,

$$\sigma(x_i^m \otimes x_j^n) = \begin{cases} \delta_j^i \delta_N^{m+n} fs(z_i)) & \text{, if } i \leq j \\ \delta_n^m n!_q fs(z_{21}^n) & \text{, if } i = 2 > j = 1 \end{cases}$$

for $0 \leq m, n < N$. Here is the connection to Hochschild cohomology, a result which is also applicable in a more general context. Recall first that by [GM] there is a Kunneth type isomorphism in equivariant Hochschild cohomology

$$H^2_G(B,k) \cong H^2_G(B_1,k) \oplus H^2_G(B_2,k) \oplus (H^1(B_1,k) \otimes H^1(B_2,k))_G,$$

where $B_i = k[x_i]/(x_i^N)$ are Nichols algebras of quantum lines and $H^1(B_i,k) = \mathrm{Der}(B_i,k) \cong \mathrm{Hom}(B_i^+/(B_i^+)^2,k)$. This means that every $\zeta \in H^2_G(B,k)$ has a unique decomposition of the form $\zeta = \zeta_1 + \zeta_2 + \zeta_{21}$. The following result has also been obtained recently with somewhat different methods in [ABM], section 5.

Theorem 3.3. *For any quantum plane the diagram*

$$\begin{array}{ccc} \mathrm{Der}_G(K,k) & \xrightarrow{\delta_{Hoch}} & H^2_G(B,k) \\ {\scriptstyle \exp}\big\downarrow & & {\scriptstyle \mathrm{Exp}_q}\big\downarrow \\ \mathrm{Alg}_G(K,k) & \xrightarrow{\delta} & \mathcal{H}^2_G(B,k) \end{array}$$

commutes if $\exp(d) = e^d$ *and* $\mathrm{Exp}_q(\zeta) = e_q^{\zeta_1} * e_q^{\zeta_2} * e_q^{\zeta_{21}}$, *where* $e^d = \sum_{n \geq 0} \frac{d^n}{n!}$ *and* $\exp_q(\xi) = e_q^\xi = \sum_{n\geq 0} \frac{\xi^n}{n!_q}$ *are the convolution exponential and q-exponential, respectively.*

Proof. It is clear that $\exp : \mathrm{Der}_G(K,k) \to \mathrm{Alg}_G(K,k)$, given by the convolution power series $\exp(d) = e^d = \sum_{n\geq 0} \frac{d^n}{n!}$, is an isomorphism of abelian groups, since the Hopf algebra $K = k[z_1, z_2, z_{21}]$ is a polynomial algebra. By [GM] there is a Kunneth type isomorphism in equivariant Hochschild cohomology

$$H^2_G(B,k) \cong H^2_G(B_1,k) \oplus H^2_G(B_2,k) \oplus (H^1(B_1,k) \otimes H^1(B_2,k))_G,$$

where $B_i = k[x_i]/(x_i^N)$ are Nichols algebras of quantum lines and $H^1(B_i,k) = \mathrm{Der}(B_i,k) \cong \mathrm{Hom}(B_i^+/(B_i^+)^2,k)$. The connecting map $\delta_{Hoch} : \mathrm{Der}_G(K,k) \to H^2_G(B,k)$ is an isomorphism, since $\mathrm{Der}_G(R,k) = 0$ and simce $\dim \mathrm{Der}_G(K,k) = 3 = \dim H^2_G(B,k)$. The connecting map $\delta : \mathrm{Alg}_G(K,k) \to \mathcal{H}^2_G(B,k)$ is injective, as mentioned above, since $\mathrm{Alg}_G(R,k) = \{\varepsilon\}$. Moreover, every element $d \in \mathrm{Der}_G(K,k)$ has a unique expression of the form $d = d_1 + d_2 + d_{21}$, every $f \in \mathrm{Alg}_G(K,k)$ is uniquely of the form $f_1 * f_2 * f_{21}$, where the notation is self explanatory, and $e^d = e^{d_1} * e^{d_2} * e^{d_{21}}$. For a general $f = f_1 * f_2 * f_{21} \in \mathrm{Alg}_G(K,k)$

one obtains the formula

$$\begin{aligned}\delta f(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_0^k \delta_0^l \delta_n^m n!_q fs(z_{21})^n \\ &+ \delta_0^k \delta_n^{m+l-N} n!_q \binom{m}{n}_q fs(z_2) fs(z_{21})^n \\ &+ \delta_0^l \delta_{k+n-N}^m m!_q \binom{n}{m}_q fs(z_1) fs(z_{21})^m \\ &+ \delta_{k+n}^{m+l} \delta_r^{k+n-N} q^{(N-l)(N-k)} r!_q \binom{m}{r}_q \binom{n}{r}_q fs(z_1) fs(z_2) fs(z_{21})^r \\ &= \delta f_1 * \delta f_2 * \delta f_{21}(x_1^k x_2^m \otimes x_1^n x_2^l),\end{aligned}$$

where $\delta f = \delta f_1 * \delta f_2 * \delta f_{21}$ also follows directly from the fact that the cofaces $\partial^i : \mathrm{Reg}_G(R,k) \to \mathrm{Reg}_G(R \otimes R, k)$ are algebra maps and that $\partial(fu)(v \otimes v) = \partial^1(fu)(v \otimes v)$. This formula shows in particular that

$$\begin{aligned}\delta e^{d_1}(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_0^m \delta_0^l d_1 s(z_1) \\ \delta e^{d_2}(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_0^k \delta_0^n d_2 s(z_2) \\ \delta e^{d_{21}}(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_0^k \delta_0^l \delta_n^m n!_q d_{21} s(z_{21})^n\end{aligned}$$

On the other hand, drawing on Lemma 3.1 again, for $d = d_1, d_2, d_{21}$ compute

$$e_q^{\delta_{Hoch} d} = e_q^{dsum} = \sum_{t \geq 0} \frac{(dsum)^t}{t!_q}$$

by evaluating the convolution powers $(dsum)^t(x_1^k x_2^m \otimes x_1^n x_2^l)$ for $t > 0$ to get

$$\begin{aligned}(d_1 sum)^t(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_1^s \delta_0^m \delta_0^l \delta_N^{k+n} d_1 s(z_1) \\ (d_2 sum)^t(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_1^s \delta_0^k \delta_0^n \delta_N^{m+l} d_2 s(z_2) \\ (d_{21} sum)^t(x_1^k x_2^m \otimes x_1^n x_2^l) &= \delta_0^k \delta_0^l \delta_n^m \delta_n^t (n!_q)^2 (d_{21} s(z_{21}))^n\end{aligned}$$

and therefore $e_q^{\delta_{Hoch} d} = \delta e^d$ for the specified derivations. This means that the map $\mathrm{Exp}_q : H_G^2(B,k) \to \mathcal{H}_G^2(B,k)$ is given by $\mathrm{Exp}_q(\zeta) = e_q^{\zeta_1} * e_q^{\zeta_2} * e_q^{\zeta_{21}}$. □

Remark: Observe that in general δf_1 and δf_2 do not commute with δf_{21}, since for example $\delta f_2 * \delta f_{21}(x_2^2 \otimes x_1 x_2^{N-1}) = q^{-1}(1+q) f_2(z_2) f_{21}(z_{21})$ and $\delta f_{21} * \delta f_2(x_2^2 \otimes x_1 x_2^{N-1}) = (1+q) f_{21}(z_{21}) f_2(z_2)$, so that in general $Exp_q(\zeta) \neq e_q^\zeta$ (see also [ABM]). But, if $\zeta_{21} = 0$, that is $f_{21} = \varepsilon$, then $\mathrm{Exp}_q(\zeta) = e_q^\zeta$, since $\delta f_2 * \delta f_1 = \delta f_1 * \delta f_2$, a result already obtained in [GM].

3.3. **Linking.** Let $V = kx_1 \oplus \ldots \oplus kx_\theta$ be any special diagram of finite Cartan type, and suppose that $i < j$ is a linkable pair, i.e: $\chi_i \chi_j = \varepsilon$. Then i and j are in different components of the Dynkin diagram, and they are not linkable to any other vertices. Let $B = TV/I$ be the Nichols albebra of V, where I is the ideal generated by the usual set S. If $S_{ij} = S \setminus \{z_{ji}\}$, where $z_{ji} = [x_j, x_i]$, then the ideal I_{ij} in TV generated by S_{ij} is still a Hopf ideal and $R_{ij} = TV/I_{ij}$ is a braided Hopf

algebra. The kernel of the canonical projection $\pi : R_{ij} \to B$ is the ideal generated by z_{ji}, which is a Hopf ideal, since z_{ji} is primitive. If K_{ij} is the Hopf subalgebra of R_{ij} generated by z_{ji} then

$$\begin{array}{ccc} K_{ij} & \xrightarrow{\kappa} & R_{ij} \\ {\scriptstyle\varepsilon}\downarrow & & {\scriptstyle\pi}\downarrow \\ k & \xrightarrow{\iota} & B \end{array}$$

is a pushout square. Moreover, as a vector space $R_{ij} \cong (B \otimes K_{ij}) \oplus Jij$, where J_{ij} is the ideal generated by the set $\{[x_k, z_{ji}] | 1 \leq k \leq \theta\}$, which is not a Hopf ideal.

Proposition 3.4. *For any special diagram of finite Cartan type and any linkable pair of verices $i < j$ in its Dykin diagram, the linear map $u : R_{ij} \to K_{ij}$, given by $u(x^a \otimes z_{ji}^n + J_{ij}) = \varepsilon(x^a) z_{ij}^n$, is a K_{ij}-bimodule coalgebra retraction for the inclusion $\kappa : K_{ij} \to R_{ij}$.*

Proof. It is clear that the $u : R_{ij} \to K_{ij}$ just defined is a linear map satisfying $u\kappa = 1_{K_{ij}}$. It is a K_{ij}-bimodule map, since in R_{ij} the element $[x^a, z_{ji}]$ is in J_{ij} for every $x^a \in B$. It is a coalgebra map, since $(B^+ \otimes K_{ij}) \oplus J_{ij}$is a coideal in R_{ij}. □

Our Theorem 3.1 and the corresponding result for Hochschild cohomology are therefore applicable. Since, as an algebra, R_{ij} is generated by the set $\{x_l | 1 \leq l \leq \theta\}$, and since the $\chi_l(x_l)$ are non-trivial roots of unity, we conclude that $\mathrm{Der}_G(R_{ij}, k) = 0$ and $\mathrm{Alg}_G(R_{ij}, k) = \{\varepsilon\}$. Moreover, for the polynomial Hopf algebra $K_{ij} = k[z_{ji}]$, the convolutian exponential map $\exp : \mathrm{Der}_G(K_{ij}, k) \to \mathrm{Alg}_G(K_{ij}, k)$ is an isomorphism of groups, and the diagram

$$\begin{array}{ccc} \mathrm{Der}_G(k_{ij}, k) & \xrightarrow{\delta_{Hoch}} & H^2_G(B, k) \\ {\scriptstyle\exp}\downarrow & & \\ \mathrm{Alg}_G(K_{ij}, k) & \xrightarrow{\delta} & \mathcal{H}^2_G(B, k) \end{array}$$

carries some information. In this generality there is no obvious map relating $H^2_G(B, k)$ to $\mathcal{H}^2_G(B, k)$, but the diagram relates the image of δ_{Hoch} to $\mathcal{H}^2_G(B, k)$. More precisely, by the Kunneth formula for the equivariant Hochschild cohomology of Nichols algebras, $\mathrm{im}\, \delta_{Hoch} \subseteq (\mathrm{Der}(B_i, k) \otimes \mathrm{Der}(B_j, k))_G$, where B_i and B_j are the Nichols algebras of the components of the Dynkin diagram containing the vertices i and j, respectively.

Corollary 3.5. *Let $i < j$ be a linkable pair of vertices in a special diagram of finite Cartan type. For the derivation $d \in \mathrm{Der}_G(K_{ij}, k)$ the Hochschild cocycle representing $\zeta = \delta_{Hoch} d \in H^2_G(B, k)$ is given by $\zeta(x^a \otimes x^b) = \delta^a_{e_j} \delta^b_{e_i} d(z_{21})$ and $e^\zeta_q = \delta e^d \in \mathcal{H}^2_G(B, k)$*

Proof. Replacing the pair $(1, 2)$ by (i, j), Lemma 3.1 holds for any linkable pair $i < j$ in any special diagram of finite Cartan type. It shows that the Hochschild

cocycle $\zeta = \delta_{Hoch}$ is of the form specified. Together with arguments, similar to those used in Theorem 3.3, it also shows that $\delta e^d = e^\zeta_q$. □

3.4. **Type** $A_1 \times \ldots \times A_1$**.** The general quantum linear space $V = kx_1 \oplus \ldots \oplus kx_\theta$ of dimension θ has G-coaction $\delta(x_i) = g_i \otimes x_i$ and G-action $gx_i = \chi_i(g)x_i$, where $\chi_i^{N_i} = \varepsilon$ and $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \neq j$.

A vertex i is linkable to at most one other vertex, since the order N_i of $q_{ii} = \chi_i(g_i)$ is supposed to be greater than 2. The vertex set $\{1, 2, \ldots, \theta\}$ can therefore be decomposed into a set L of linkable pairs of the form $i < j$ and a set of non-linkable singletons $L^\perp$, and it can be ordered accordingly. A quantum linear space is therefore a collection of quantum planes together with a bunch of quantum lines with pushout squares

$$\begin{array}{ccccccc} K_{ij} & \xrightarrow{\kappa_{ij}} & R_{ij} & , & K_l & \xrightarrow{\kappa_l} & R_l \\ \varepsilon\downarrow & & \pi_{ij}\downarrow & & \varepsilon\downarrow & & \pi_l\downarrow \\ k & \xrightarrow{\iota} & B_{ij} & & k & \xrightarrow{\iota} & B_i \end{array}$$

for $(i, j) \in L$ and $l \in L^\perp$, respectively. The braided tensor product of all these squares represents the Nichols algebra of the quantum linear space. The following considerations about such braided tensor products together with the results for $\theta \leq 2$ will describe the deforming cocycles for all quantum linear spaces.

If a subset S of $\{1, 2, \ldots, \theta\}$ is such that none of its vertices is linkable to any vertex not in S then the complement T has the same property and $\{1, 2, \ldots, \theta\} = S \cup T$. The elements of K_S commute with the elements of R_T and the elements of K_T commute with those of R_S. It follows that there is a commutative diagram of coalgebras

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & R & \xrightarrow{\pi} & B \\ \rho_K\downarrow & & \rho_R\downarrow & & \rho_B\downarrow \\ K_S \otimes K_T & \xrightarrow{\kappa_S \otimes \kappa_T} & R_S \otimes R_T & \xrightarrow{\pi_S \otimes \pi_T} & B_S \otimes B_T \end{array}$$

with $\rho = (p_S \otimes p_T)\Delta$ is an isomorphism with inverse $\rho^{-1} = m(i_S \otimes i_T)$. The projections $e_S = i_S p_s$ and $e_T = i_T p_T$ on K, R and B have the property that

$$e_S * e_T = \rho^{-1}\rho = 1\ ,\ ue_S = e_S u\ ,\ ue_T = e_T u\ ,\ u = e_S u * e_T u = ue_S * ue_T$$

and, moreover, since the elments of K_S commute with those of K_T, also $e_T * e_S = e_S * e_T = 1_K$ on K. The latter is of course not true on$R$ and B, because $e_T * e_S(x^a_S x^b_T) = \chi^a(g^b)e_S * e_T(x^a x^b$. With $u_S = p_S u i_S$ and $u_T = p_T u i_T$ the

diagram

$$\begin{array}{ccccccccc} R & \xrightarrow{p_S} & R_S & \xrightarrow{i_S} & R & \xleftarrow[i_T]{} & R_T & \xleftarrow[p_T]{} & R \\ {\scriptstyle u}\downarrow & & {\scriptstyle u_S}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle u_T}\downarrow & & {\scriptstyle u}\downarrow \\ K & \xrightarrow{p_S} & K_S & \xrightarrow{i_S} & K & \xleftarrow[i_T]{} & K_T & \xleftarrow[p_T]{} & K \end{array}$$

commutes. The projections $e_S = i_S\pi_S$ and $e_T = i_T\pi_T$ on R and K satisfy

$$e_Su = ue_S\ ,\ e_Tu = ue_T\ ,\ e_S * e_T = \rho_S^{-1}\rho_S = 1\ ,\ u = ue_S * ue_T = e_Su * e_Tu$$

and the diagram

$$\begin{array}{ccc} R & \xrightarrow{u} & K \\ {\scriptstyle \rho_R}\downarrow & & {\scriptstyle \rho_K}\downarrow \\ R_S \otimes R_T & \xrightarrow{u_S\otimes u_T} & K_S \otimes K_T \end{array}$$

commutes. Moreover, since the elements of K_S and K_T commute, we have $e_T*e_S = e_S * e_T = 1_K$ on K. This is of course not the case on R or on B, because $e_T * e_S(x^ax^{a'}z^bz^{b'}) = \chi^a(g^{a'})e_S * e_T(x^ax^{a'}z^bz^{b'})$.

Proposition 3.6. *Suppose that $\{1, 2, \dots, \theta\} = S \cup T$ is such that none of the vertices of S is linkable to any vertex of T, then $u = u_S * u_T : R \to K$, where $u_S = ue_S = e_Su$ and $u_T = ue_T = e_Tu$ and the square*

$$\begin{array}{ccc} \mathrm{Alg}_G(K_S, k) \times \mathrm{Alg}_G(K_T, k) & \xrightarrow{\delta} & \mathcal{H}^2_G(B_S, k) \times \mathcal{H}^2_G(B_T, k) \\ {\scriptstyle \rho^1}\downarrow & & {\scriptstyle \rho^2}\downarrow \\ \mathrm{Alg}_G(K, k) & \xrightarrow{\delta} & \mathcal{H}^2_G(B, k) \end{array}$$

commutes, where $\rho^1(f, f') = (f \otimes f')\rho$ and $\rho^2(\sigma, \sigma') = (\sigma \otimes \sigma')(1 \otimes c \otimes 1)(\rho \otimes \rho)$. Moreover, ρ^1 is an isomorphism, while ρ^2 is injective.

Proof. With our assumptions and $f \in \mathrm{Alg}_G(K, k)$ we have $f = f(e_S * e_T) = fe_S * fe_T$ and$fu = fe_Su * fe_Tu$. Observe that the inverse of ρ^1 is given by $(\rho^1)^{-1}(f) = (fi_S, fi_T)$:

$$\rho^1(\rho^1)^{-1}(f) = (fi_S \otimes fi_T)\rho = f(e_S * e_T) = f,$$

while

$$(\rho^1)^{-1}\rho^1(f_S, f_T) = ((f_S\otimes f_T)\rho i_S, (f_S\otimes f_T)\rho i_T) = (f_S\otimes\varepsilon)\Delta, \varepsilon\otimes f_T)\Delta) = (f_S, f_T),$$

since $\rho i_S = (1 \otimes \iota\varepsilon)\Delta$ and $\rho i_T = (\iota\varepsilon \otimes 1)\Delta$. A similar argument shows that ρ^2 has a left inverse ψ given by $\psi(\sigma) = (\sigma(i_S \otimes i_S), \sigma(i_T \otimes i_T))$.

The diagram commutes, because

$$\begin{aligned} \partial^i(\rho^1(f_S, f_T)u) &= \partial^i((f_S \otimes f_T)\rho u) = \partial^i(f_Su_Sp_S * f_Tu_Tp_T) \\ &= \partial^i(f_Su_Sp_S) * \partial^i(f_Tu_Tp_T) = (\partial^i f_Su_S \otimes \partial^i f_Tu_T)\rho_{R\otimes R} \\ &= \theta^2(\partial^i f_Su_S, \partial^i f_Tu_T), \end{aligned}$$

where we used $d^i(p_S\otimes p_S)=p_Sd^i$, $d^i(p_T\otimes p_T)=p_Td^i$ and $\rho_{R\otimes R}=(1\otimes c\otimes 1)(\rho\otimes\rho)=(1\otimes c\otimes 1)(p_S\otimes p_T\otimes p_S\otimes p_T)(\Delta_R\otimes\Delta_R)=(p_S\otimes p_S\otimes p_T\otimes p_T)\Delta_{R\otimes R}$. Moreover,

$$\begin{aligned}\partial^i fu &= \partial^i(fe_S*fe_T)u=\partial^i(fe_Su*fe_Tu)=\partial^i fe_Su*\partial^i fe_Tu\\ &= \partial^i fi_Su_Sp_S*\partial^i fi_Tu_Tp_T=(\partial^i fi_Su_S\otimes\partial^i fi_Tu_T)\rho_{R\otimes R}\end{aligned}$$

as well.

Since $e_T*e_S=e_S*e_T=1_K$ on K and hence $\partial^i fu=\partial^i fe_Su*\partial^i fe_Tu=\partial^i fe_Tu*\partial^i fe_Su$, and since $\partial fu(v\otimes v)=\partial^1 fsu(v\otimes v)$, it follows that

$$\partial fu=\partial fe_Su*\partial fe_Tu$$

as required. □

A comparison with Hochschild cohomology can be obtained inductively via a generalized 'exponential' map, making use of the isomorphism

$$\begin{tikzcd}
\mathrm{Der}_G(K,k) \arrow[r, "\delta_{hoch}"] \arrow[d, "\cong"'] & H^2_G(B,k) \arrow[d, "\cong"'] \\
\mathrm{Der}_G(K_S,k)\oplus\mathrm{Der}_G(K_T,k) \arrow[r, "\delta_{hoch}\oplus\delta_{hoch}"] & H^2_G(B,k)
\end{tikzcd}$$

and Proposition 3.6 to get a commutative square

$$\begin{tikzcd}
\mathrm{Der}_G(K,k) \arrow[r, "\delta_{hoch}"] \arrow[d, "\exp"'] & H^2_G(B,k) \arrow[d, "\mathrm{Exp}"'] \\
\mathrm{Alg}_G(K,k) \arrow[r, "\delta"] & \mathcal{H}^2_G(B,k)
\end{tikzcd}$$

which says that

$$\delta e^{d_S+d_T}=\delta e^{d_S}*\delta e^{d_T}=\mathrm{Exp}_S(\delta_{hoch}d_S)*\mathrm{Exp}_T(\delta_{hoch}d_T)=\mathrm{Exp}(\delta_{hoch}(d_S+d_T))$$

by extending the notation naturally. In particular, if $f\in\mathrm{Alg}_G(K,k)$ and $\sigma=\delta f$ then

$$\sigma(x_i^m\otimes x_j^n)=fsu(x_i^mx_j^n)=\begin{cases} fsu(z_i) & \text{, if } i=j \text{ and } m+n=N_i\\ n!_{q_i}fs(z_{ji})^n & \text{, if } i>j \text{ linkable and } m=n\\ 0 & \text{, otherwise.}\end{cases}$$

Remark. Observe that Proposition 3.6 holds for any special diagram of finite Cartan type, provided that a K-bimodule coalgebra retraction $u:R\to K$ exists. This is because $\partial^i(fe_Su)*\partial^j(fe_Tu)=\partial^j(fe_Tu)*\partial^i(fe_Su)$ for $f\in\mathrm{Alg}_G(K,k)$ and $i\le j$ if S and T are not linkable.

3.5. **The connected case.** Let $\mathcal{D}$ be a special connected datum of finite Cartan type with Cartan matrix (a_{ij}). The vector space $V = V(\mathcal{D})$ can also be viewed as a crossed module in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$, where $\mathbf{Z}[I]$ is the free abelian group on the set of simple roots $I = \{\alpha_1, \dots, \alpha_\theta\}$. The $\mathbf{Z}[I]$-degree of a word $x = x_{i_1}x_{i_2}\dots x_{i_n}$ in the tensor algebra $mathcala(V)$ is defined by $\deg(x) = \sum_{i=1}^{\theta} n_i\alpha_i$, where n_i is the number of ocurrencies of x_i in x. The Weyl group $W \subset \mathrm{Aut}(\mathbf{Z}[I])$ is generated by the automorphisms s_i defined by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. The root system $\Phi = \cup_{i=1}^{\theta} W(\alpha_i)$ is the union of the orbits of simple roots in $[I]$, and

$$\Phi^+ = \{\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \Phi | n_i \geq 0\}$$

is the set of positive roots. The Hopf algebra $\mathcal{A}(V)$, the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}}x_i(x_j)|1 \leq i \neq j \leq \theta)$ and its Hopf subalgebra $K(\mathcal{D})$ generated by $\left\{ x_\alpha^N \middle| \alpha \in \Phi^+ \right\}$, as well as the Nichols algebra $B(V) = R(V)/(x_\alpha^N)$, are all Hopf algebras in ${}^{\mathbf{Z}[I]}_{\mathbf{Z}[I]}YD$. In particular, their comultiplications are $\mathbf{Z}[I]$-graded. By construction, for $\alpha \in \Phi^+$, the root vector x_α is $\mathbf{Z}[I]$-homogeneous of $\mathbf{Z}[I]$-degree α, so that $\delta(x_\alpha) = g_\alpha \otimes x_\alpha$ and $gx_\alpha = \chi_\alpha(g)x_\alpha$. For $1 \leq l \leq p$ and for $a = (a_1, a_2, \dots, a_p) \in \mathbf{N}^p$ write $\underline{a} = \sum_{i=1}^{p} a_i\beta_i$ and

$$g^a = g_1^{a_1}g_2^{a_2}\dots g_p^{a_p} \in G \ , \ \chi^a = \chi_1^{a_1}\chi_2^{a_2}\dots\chi_p^{a_p} \in \tilde{G} \ , \ x^a = x_{\beta_1}^{a_1}x_{\beta_2}^{a_2}\dots x_{\beta_p}^{a_p} \in R(\mathcal{D}).$$

In particular, for $e_l = (\delta_{kl})_{1\leq k\leq p}$, where δ_{kl} is the Kronecker symbol, $\underline{e}_l = \beta_l$ and $x^{e_l} = x_{\beta_l}$ and $x^{Ne_l} = x_{\beta_l}^N = z_l$ for $1 \leq l \leq p$. In this notation

$$\{x^a|0 \leq a_i\} \ , \ \{z^b|0 \leq b_i\} \ , \ \{x^a|0 \leq a_i < N\}$$

form a PBW-basis for $R(V)$, $K(V)$ and $B(V)$, respectively. The height of $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i \in \mathbf{Z}[I]$ is defined to be the integer $ht(\alpha) = \sum_{i=1}^{\theta} n_i$. Observe that if $a, b, c \in \mathbf{N}^p$ and $\underline{a} = \underline{b} + \underline{c}$ then

$$g^a = g^bg^c \ , \ \chi^a = \chi^b\chi^c \text{ and } \mathrm{ht}(\underline{b}) < \mathrm{ht}(\underline{a}) \text{ if } \underline{c} \neq 0.$$

By [AS] (Theorem 2.6), the sets

$$\{z^b|0 \leq b_i\} \ , \ \{x^az^b|0 \leq a_i < N, 0 \leq b_j\} \ , \ \{x^a|0 \leq a_i < N\}$$

form a basis for $K(V)$, $R(V)$ and $B(V)$, respectively. The squares

$$\begin{array}{ccccccc} K(V) & \xrightarrow{\kappa} & R(V) & , & K\#kG & \xrightarrow{\kappa\#1} & R\#kG \\ \varepsilon\downarrow & & \pi\downarrow & & \varepsilon\#1\downarrow & & \pi\#1\downarrow \\ k & \xrightarrow{\iota} & B(V) & & kG & \xrightarrow{\iota\#1} & B\#kG \end{array}$$

are pushout squares of braided Hopf algebras and their bosonizations, respectively.

Moreover, the K-module isomorphism $\vartheta : R \to B \otimes K$ given by $\vartheta(x^az^b) = x^a \otimes z^b$, can be used to get a K-module retraction $u = (\varepsilon \otimes 1)\vartheta : R(V) \to K(V)$,

$u(x^a z^b) = \varepsilon(x^a)z^b)$, for the inclusion of $\kappa : K(V) \to R(V)$. Thus, the conditions for the 5-term sequence in Hochschild cohomology are satisfied. The connecting map

$$\delta_{hoch} : \mathrm{Der}_G(K,k) \to H^2_G(B,k)$$

which is injective since $\mathrm{Der}_G(R,k) = 0$, is such that $\delta_{hoch} d(\pi \otimes \pi) = \partial_{hoch}(du) = -dum_R$, where $\pi : R(V) \to B(V)$ is the canonical projection. The K-module map $u : R \to K$ just defined is not a coalgebra map in general.

Observe that $K = k[z_\alpha | \alpha \in \Phi^+]$ is a polynomial algebra, since by our assumption $\chi_i^N = \varepsilon$ for $1 \le i \le \theta$. The algebra isomorphism $\rho : \oplus_{\alpha\in\Phi^+} K_\alpha \to K$, given by $\rho(z^{n_1}_{\alpha_1} \otimes z^{n_2}_{\alpha_2} \otimes \ldots \otimes z^{n_p}_{\alpha_p}) = z^{n_1}_{\alpha_1} z^{n_2}_{\alpha_2} \ldots z^{n_p}_{\alpha_p}$, induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Der}_G(K,k) & \xrightarrow{\rho_{\mathrm{Der}}} & \oplus_{\alpha\in\Phi^+} \mathrm{Der}_G(K_\alpha,k) \\ {\scriptstyle \mathrm{Exp}}\big\downarrow & & {\scriptstyle \exp}\big\downarrow \\ \mathrm{Alg}_G(K,k) & \xrightarrow{\rho_{\mathrm{Alg}}} & \times_{\alpha\in\Phi^+} \mathrm{Alg}_G(K_\alpha,k) \end{array}$$

of sets, with $\rho_{\mathrm{Der}}(d) = (di_\alpha)$, $\rho_{\mathrm{Alg}}(f) = (fi_\alpha)$, $\exp((d_\alpha)) = (e^{d_\alpha})$ and $\mathrm{Exp}(d) = \rho^{-1}_{\mathrm{Alg}} \exp \rho_{\mathrm{Der}}$, where $i_\alpha : K_\alpha \to K$ and $p_\alpha ; K \to K_\alpha$ are the obvious canonical injections and projections. This means more explicitly that

$$\mathrm{Exp}(d)(z^{n_1}_{\alpha_1} z^{n_2}_{\alpha_2} \ldots z^{n_p}_{\alpha_p}) = e^{di_1}(z^{n_1}_{\alpha_1}) e^{di_2}(z^{n_2}_{\alpha_2}) \ldots e^{di_p}(z^{n_p}_{\alpha_p})$$

for $d \in \mathrm{Der}_G(K,k)$.

If $\kappa : K \to R$ has a K-module coalgebra retraction $u_\infty : R \to K$ then Theorem 2.1 is applicable, and the diagram

$$\begin{array}{ccc} \mathrm{Der}_G(K,k) & \xrightarrow{\delta_{hoch}} & H^2_G(B,k) \\ {\scriptstyle \mathrm{Exp}}\big\downarrow & & \\ \mathrm{Alg}_G(K,k) & \xrightarrow{\delta} & \mathcal{H}^2_G(B,k) \end{array}$$

connects the relevant part of the Hochschild cohomology $H^2_G(B,k)$ to the multiplicative cohomology $\mathcal{H}^2_G(B,k)$.

Proposition 3.7. *Let V be a special (connected) diagram of finite Cartan type. If $K(V)$ is a K-module coalgebra retract in $R(V)$ then the connecting map*

$$\delta : \mathrm{Alg}_G(K,k) \to \mathcal{H}^2_G(B,k)$$

is injective.

Proof. The simple root vectors x_α, where $\alpha \in \Phi^+$ is a simple root, generate R as an algebra. Moreover, $f(x_\alpha) = f(gx_\alpha) = \chi_\alpha(g)f(x_\alpha)$ for every $g \in G$ and every $f \in \mathrm{Alg}_G(R,k)$. It follows that $\mathrm{Alg}_G(R,k) = \{\varepsilon\}$, since $q_\alpha = \chi_\alpha(g_\alpha)$ is a non-trivial root of unity for every simple root $\alpha \in \Phi^+$.

Now suppose that $\delta f = \delta f'$ in $\mathcal{H}^2_G(B,k)$ for some $f, f' \in \mathrm{Alg}_G(K,k)$. The representing cocycles σ and σ' are equivalent, so that $\sigma' = \partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}$ for some $\chi \in \mathrm{Reg}_G(B,k)$. It follows that

$$\begin{aligned}
\partial^0 f'u * \partial^2 f'u * \partial^2 f'su &= \partial f'u = (\pi\otimes\pi)^*\sigma' = (\pi\otimes\pi)^*(\partial^0\chi * \partial^2\chi * \sigma * \partial^1\chi^{-1}) \\
&= \partial^0(\pi^*\chi) * \partial^2(\pi^*\chi) * \partial fu * \partial^1(\pi^*\chi^{-1}) \\
&= \partial^0(\pi^*\chi) * \partial^2(\pi^*\chi) * \partial^0 fu * \partial^2 fu * \partial^1 fsu * \partial^1(\pi^*\chi^{-1}) \\
&= \partial^0(\pi^*\chi) * \partial^0 fu * \partial^2(\pi^*\chi) * \partial^2 fu * \partial^1 fsu * \partial^1(\pi^*\chi^{-1}) \\
&= \partial^0(\pi^*\chi * fu) * \partial^2(\pi^*\chi * fu) * \partial^1(fsu * \pi^*\chi^{-1})
\end{aligned}$$

since the $\mathrm{im}\,\partial^0$ and $\mathrm{im}\,\partial^2$ commute elementwise. so that $\partial^2(\pi^*\chi) * \partial^0 fu = (\pi^*\chi \otimes \varepsilon\otimes\varepsilon\otimes fu)\Delta_{R\otimes R} = (\varepsilon\otimes fu\otimes\pi^*\chi\otimes\varepsilon)\Delta_{R\otimes R} = \partial^0 fu * \partial^2(\pi^*\chi)$. This means, again using the elementwise commutativity of $\mathrm{im}\,\partial^0$ and $\mathrm{im}\,\partial^2$, that

$$\begin{aligned}
\partial^1(fsu * \pi^*\chi^{-1} * f'u) &= \partial^1(fsu * \pi^*\chi^{-1}) * \partial^1 f'u \\
&= \partial^2(\pi^*\chi * fu)^{-1} * \partial^0(\pi^8\chi * fu)^{-1}\partial^o f'u * \partial^2 f'u \\
&= \partial^0(fsu * \pi * \chi^{-1})\partial^0 f'u\partial^2(fsu * \pi^*\chi^{-1})\partial^2 f'u \\
&= \partial^0(fsu * \pi^*\chi^{-1} * f'u) * \partial^2(fsu * \pi^*\chi^{-1} * f'u)
\end{aligned}$$

so that $fsu * \pi^*\chi^{-1} * f'u \in \mathrm{Alg}_G(R,k) = \{\varepsilon\}$ and then $f'u = \pi^*\chi * fu$. But then

$$f' = f'u\kappa = (\pi^*\chi * fu)\kappa = fu\kappa * \chi\pi\kappa = \chi\varepsilon * f = \varepsilon * f = f$$

as required. $\square$

The multiplicative cocycle σ representing the cohomology class δf is given by

$$(\pi\otimes\pi)^*\sigma = \partial fu_\infty = \partial^0 fu_\infty * \partial^2 fu_\infty * \partial^1 fsu_\infty = (fu_\infty \otimes fu_\infty) * fsu_\infty m_R$$

or, equivalently $\sigma = \partial fu_\infty(v\otimes v)$, where $v : B \to R$ is the obvious linear section of the canonical projection $\pi : R \to B$.

Conjecture 1. *For every special connected diagram of finite Cartan type V the braided Hopf subalgebra K is a K-module coalgebra retract in R.*

Here is a recursive procedure to verify the conjecture. Let B_i be the linear span in B of all ordered words involving root vectors of height $\leq i$ only. For $i > 1$ let $B_i^j \subset B_i$ be the linear span of all ordered monomials in B_i containing at most j distinct root vectors of height i. Then B_i^j is a subcoalgebra of B. The inclusion $v_i^j : B_i^j \to R$ is not a coalgebra map, but $B_i^j \otimes K \subset R$ is a subcoalgebra under the coalgebra structure inherited from R (not the tensor product coalgebra structure). This gives a finite filtration $B_i \subseteq B_{i+1}^j \subseteq B_{i+1}$ of B and $\cup_{i\geq 0} B_i = B$. Observe that $B_i^0 = B_{i-1}$ and $B_i^j = B_i$ for some j.

- For B_1 let $u_1 = \varepsilon\otimes 1 : B_1\otimes K \to K$, which is a coalgebra map.

- Suppose a coalgebra retraction $u^j_{i+1} = m_K(\varphi^j_{i+1} \otimes 1) : B^j_{i+1} \otimes K \to K$ has been constructed. Extend φ^j_{i+1} linearly to B^{j+1}_{i+1} by sending to zero all PBW-monomials involving more than j distinct root vectors of height $i+1$. For such a PBW-monomial $x \in B^{j+1}_{i+1} \setminus B^j_{i+1}$ find a $z \in K$ such that $\Delta_K z - z \otimes 1 - 1 \otimes z = (u^j_{i+1} \otimes u^j_{i+1})\Delta_R v^{j+1}_{i+1} x$. Now define $\varphi^{j+1}_{i+1} : B^{j+1}_{i+1} \to K$ by $\varphi^{j+1}_{i+1}(x) = z$ and $\varphi^{j+1}_{i+1}|_{B^j_{i+1}} = \varphi^j_{i+1}$. Then $u^{j+1}_{i+1} = m_K(\varphi^{j+1}_{i+1} \otimes 1) : B^{j+1}_{i+1} \otimes K \to K$ is a K-module coalgebra map.
- Since B is finite dimensional $B = B^j_i$ for some pair (i,j). Then $u_\infty = u^j_i = m_K(\varphi^j_i \otimes 1)\vartheta : R \to B \otimes K \to K$ is a retraction for the inclusion $\kappa : K \to R$.

3.6. **Type A_2.** Here we have a crossed kG-module $V = kx_1 \otimes kx_2$ with coaction $\delta(x_i) = g_i \otimes x_i$ and action $gx_i = \chi_i(g)x_i$, where $\chi_i(g_i) = q$ and $\chi_j(g_i)\chi_i(g_j) = q_{ij}q_{ji} = q^{-1}$. If $e_{12} = x_1$, $e_{23} = x_2$ and $e_{13} = [e_{12}, e_{23}] = [x_1, x_2]$ then $\{e^m_{12}e^n_{13}e^l_{23} | 0 \le m,n,l < N\}$, $\{e^m_{12}e^n_{13}e^l_{23} | 0 \le m,n,l\}$ and $\{z^m_{12}z^n_{13}z^l_{23} | 0 \le m,n,l\}$, where $z_{ij} = e^N_{ij}$, form a basis for $B(V)$, $R(V)$ and $K(V)$, respectively. In this notation, taken from [AS1], the comultiplications in the bosonisations are determined by

$$\Delta(e_{ij}) = \sum_{i \le p \le j} \lambda_{ipj} e_{ip} g_p g_j \otimes e_{pj},$$

where $e_{ii} = 1$ and

$$\lambda_{ipj} = \begin{cases} 1 & \text{, if } i = p \text{ or } p = j \\ 1 - q^{-1} & \text{, if } i \ne p \ne j \end{cases}$$

Proposition 3.8. *For diagrams of type A_2 the Hopf subalgebra $K \subset R$ is a K-bimodule coalgebra retract, with retraction $u_\infty = u_2 : R \to K$.*

Proof. It will be necessary to deform the K-bimodule retraction $u = (\varepsilon \otimes 1)\vartheta : R \to K$ somewhat to make it a coalgebra map in this case. Observe that $u_1 = \varepsilon \otimes 1 : B_1 \otimes K \to K$ is a K-module coalgebra map. The following arguments show that its extension $u = \varepsilon \otimes 1 : B \otimes K \to K$ is not a coalgebra map. In $R\#kG$ we get

$$\begin{aligned}\Delta(e^m_{12}e^n_{13}e^l_{23}) \quad = \quad & \sum_{1 \le p_i \le 2, 1 \le q_j \le 3, 2 \le r_k \le 3} \lambda_{1p_12} \dots \lambda_{1p_m2}\lambda_{1q_13} \dots \lambda_{1q_n3}\lambda_{2r_13} \dots \lambda_{2r_l3} \\ & e_{1p_1}g_{p_12} \dots e_{1p_m}g_{p_m2}e_{1q_1} \dots e_{1q_n}g_{q_n3}e_{2r_1}g_{r_13} \dots e_{2r_l}g_{r_l3} \\ & \otimes e_{p_12} \dots e_{p_m2}e_{q_13} \dots e_{q_n3}e_{r_13} \dots e_{r_l3}\end{aligned}$$

which contains the term $\lambda^n_{123}\chi^{\binom{n}{2}}_{12}(g_{23})e^{m+n}_{12}g^{n+l}_{23} \otimes e^{n+l}_{23}$, the only term that may make $(u \otimes u)\Delta(e^m_{12}e^n_{13}e^l_{23}) \ne 0$. In particular, if $m + n = N = n + l$ then $m = l$ and this term

$$\lambda^n_{123}\chi^{\binom{n}{2}}_{12}(g_{23})e^N_{12}g^N_{23} \otimes e^N_{23}$$

is a non-zero element in $(K\#kG)\otimes(K\#kG)$. It follows directly that

$$\begin{aligned}(u\otimes u)\Delta(e_{12}^m e_{13}^n e_{23}^l) &= u(e_{12}^m e_{13}^n e_{23}^l)\otimes 1 + g_{12}^m g_{13}^n g_{23}^l \otimes u(e_{12}^m e_{13}^n e_{23}^l)\\ &\quad + \lambda_{123}^n \chi_{12}^{\binom{n}{2}}(g_{23})u(e_{12}^{m+n})g_{23}^{n+l}\otimes u(e_{23}^{n+l})\end{aligned}$$

for $0\le m,n,l<N$. In particular, $(u\otimes u)\Delta(e_{12}^m e_{13}^n e_{23}^l)\neq 0$ if and only if $m+n=N=n+l$, and then

$$(u\otimes u)\Delta(e_{12}^{N-n}e_{13}^n e_{23}^{N-n}) = \lambda_{123}^n \chi_{12}(g_{23})^{\binom{n}{2}} z_{12}h_{23}\otimes z_{23}$$

while $u(e_{12}^m e_{13}^n e_{23}^l)=0$. The K-bimodule retraction $u: R\to K$ defined by $u(x^a z^b)=\varepsilon(x^a)z^b$ is therefore not a coalgebra map. To remedy this situation, observe that

$$\Delta(z_{13}) = z_{13}\otimes 1 + h_{13}\otimes z_{13} + (1-q^{-1})^N \chi_{12}^{\binom{N}{2}}(g_{23})z_{12}h_{23}\otimes z_{23}$$

and define $u_2: R\to K$ by

$$u_2(e_{12}^m e_{13}^n e_{23}^l z) = \delta_l^m \delta_N^{m+n}(1-q^{-1})^{n-N}\chi_{12}(g_{23})^{\binom{n}{2}-\binom{N}{2}} z_{13}z$$

for $z\in K$. Observe that $u_2 = m_K(\varphi_2\otimes 1)\vartheta : R\to B\otimes K\to K\otimes K\to K$, where $\varphi_2: B\to K$ is given by

$$\varphi_2(e_{12}^m e_{13}^n e_{23}^l) = (1-q^{-1})^{-m}\chi_{12}(g_{23})^{\binom{n}{2}-\binom{m+n}{2}}u(e_{12}^{m-t}e_{13}^{n+t}e_{23}^{l-t}),$$

with $t=\min(m,l)$. It then follows by construction that $u_\infty = u_2 : R\to K$ is a K-bimodule coalgebra retraction for $\kappa: K\to R$. □

The connecting map $\delta:\mathrm{Alg}_G(K,k)\to\mathcal{H}_G^2(B,k)$ guaranteed by Theorem 2.1 is injective by Proposition 3.7, since $\mathrm{Alg}_G(R,k)=\{\varepsilon\}$ and since all elements of $\mathrm{im}\,\partial^0$ commute with those of $\mathrm{im}\,\partial^2$. The resulting cocycle deformations account for all liftings of $B\#kG$.

Results for type A_n, $n>2$, and type B_2 are in the pipeline. They will be a subject of a forthcoming paper.

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